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Estimate of the Time Rate of Entropy Dissipation for Systems of Conservation Laws

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A priori estimates for weak solutions of nonlinear systems of conservation laws remain in short supply. In this note we obtain an estimate of the rate of total entropy dissipation for initial/boundary value problems for such systems, of any dimension and in any number of space variables. The essential assumptions made

I. INTRODUCTION

Any number of weak solutions of a nonlinear system of conservation laws may correspond to the same given initial/boundary data. At least for some such problems, the "physical" weak solution is characterized by the maximum time rate of total entropy dissipation [1, 6, 8]. Here we obtain an a priori estimate of the rate of total entropy generation, which is independent of the spatial variation of the solution at any time after the initial value.

We consider systems equipped with a convex entropy density, thus necessarily hyperbolic [3], and for present purposes conveniently written in symmetric form [4, 5, 7],

$$\frac{\partial}{\partial t}\phi_{z_j}+\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\psi^i_{z_j}=0, \qquad j=1, ..., n; \quad x\in\Omega, \quad t>0 \tag{1.1}$$

$$z(x, t)\in B(x, t), \qquad B(x, t)\subset\mathbb{R}^n/\varnothing, \qquad x\in\partial\Omega, \quad t>0, \tag{1.2}$$

with the sets B and the initial data $z(x, 0), x\in\bar{\Omega}$ given. Here ϕ, ψ^i are smooth functions of $z=(z_l, ..., z_n)^T$, with ϕ strictly convex upwards. For simplicity, we take $\Omega\subset\mathbb{R}^m$ bounded with smooth boundary $\partial\Omega$. The dimension of the system (1.1) n, and the number of space variables m are arbitrary.

For present purposes, a weak solution of (1.1), (1.2) is any function z measurable at each $t\geqslant 0$ from Ω into some bounded set $D\subset\mathbb{R}^n$, satisfying

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(1.1, 1.2) in the sense of distributions. In particular there is no entropy condition imposed.

For any such solution, we obtain a lower bound for the total entropy generation $G(t)$, $t>0$, given by

$$G(t)=\int_{\Omega}(U(z(\cdot,t))-U(z(\cdot,0)))+\int_{\Sigma_t}F, \tag{1.3}$$

where

$$U=z\cdot\phi_z-\phi \tag{1.4}$$

is the entropy density, convex upwards in ϕ_z and with $z=\partial U/\partial(\phi_z)$ [7];

$$\Sigma_t=\partial\Omega\times(0,t); \tag{1.5}$$

$$F=v\cdot(F^1,...,F^m)^T, \tag{1.6}$$

v the outward unit vector on $\partial\Omega$ and

$$F^i=z\cdot\psi^i_z-\psi^i, \tag{1.7}$$

the corresponding entropy flux functions.

In this generality with respect to solutions, it is not clear that G is even continuous in t, so we consider primarily the quantity g.l.b. $G(t)/t$ as $t\downarrow 0$. We expect this to be nonpositive, as $G(t)=0$ for smooth solutions and $G(t)\leqslant 0$ is expected for "physical" weak solutions.

Defermos [1] conjectured that the forward time derivative of G is minimized by the physical weak solution, and showed this condition to be equivalent to the classical entropy condition for scalar equations $(n=1)$ and for the p-system $(n=2)$. However, Hsiao found a surprisingly simple counterexample, indeed a Riemann problem for the compressible Euler equations [6]. For similarity solutions of Riemann problems, $G(t)/t$ is a constant, depending on the discontinuities appearing in some weak solution z. For systems of any dimension n, it is known that this constant is bounded from below, and that for Riemann problems with small initial variation (and other structural assumptions) that Defermos' conjecture is true [8].

These results all correspond to the case of one space dimension, no boundary $\partial\Omega$, and solutions of bounded variation. Indeed, it is clear from consideration of simple examples, e.g., the Burgers equation, that a lower bounded for $G(t)/t$ must depend on the variation of the initial data, on the boundary conditions and on the region of phase space D, as otherwise arbitrarily strong shocks might enter Ω immediately from the boundary. We introduce a measure of variation of the initial data for $m\geqslant 2$ in the following section.

Three features of the present results are noted here: The estimate for g.l.b $G(t)/t$ depends on the variation of the initial data, but not on that of the solution at later time. As with the uniqueness result of Diperna [2] for sufficiently smooth solutions of (1.1, 1.2), there are no additional structural assumptions, such as that of genuine nonlinearity. Using the idea of "entropy dissipative boundary conditions," defined precisely below but also corresponding to boundary conditions compatible with the method of [2], the contribution to $G(t)$ from the boundary becomes $\geqslant -O(t^2)$.

II. PRELIMINARIES

Given $u\colon \Omega \to \mathbb{R}$ and some unit of length $\bar{\varepsilon}$, we measure the variation of u by a seminorm

$$|u|_{E(\Omega)} \equiv \sup_{0<\varepsilon\leqslant\bar{\varepsilon}} \frac{\bar{\varepsilon}}{\varepsilon} \int_{\Omega} \operatorname*{ess\,sup}_{\substack{|y-x|<\varepsilon \\ y\in\Omega}} |u(y)-u(x)|\,dx \tag{2.1}$$

and designate a linear vector space

$$E(\Omega) = \{u \mid \|u\|_{L_1(\Omega)} + |u|_{E(\Omega)} < \infty\}. \tag{2.2}$$

In one dimension, i.e., Ω an interval, $E(\Omega)$ coincides with $BV(\Omega)$. More generally:

LEMMA 1.

$$\text{The elements of } E(\Omega) \text{ are bounded in } L_\infty(\Omega). \tag{2.3}$$

$$E(\Omega) \text{ is a pointwise multiplicative ring.} \tag{2.4}$$

$$E(\Omega) \subset (\partial C_0(\Omega))^*, \quad \text{where}$$

$$\partial C_0(\Omega) = \left\{ \frac{\partial\theta}{\partial x_i} \,\middle|\, \theta \in C_0(\Omega),\, i=1, ..., m \right\}. \tag{2.5}$$

Let $\eta\colon \Omega\times\Omega \to \mathbb{R}$ be nonnegative, smooth, of unit mass, i.e., $\int_\Omega \eta(y,x)\,dx = 1$ for all $y\in\Omega$, and of compact support in $|x-y|<\delta$. For $u\in E(\Omega)$, $w(y) = \int_\omega \eta(y,x)\,u(x)\,dx$,

$$\|u-w\|_{L_1(\Omega)} \leqslant \frac{\delta}{\bar{\varepsilon}} |u|_{E(\Omega)}. \tag{2.6}$$

$$E(\Omega) \text{ is compactly contained in } L_1(\Omega). \tag{2.7}$$

The proof is deferred to Section IV.

DEFINITION. The boundary conditions (1.2) are called dissipative if for all $x \in \partial\Omega$, $t > 0$ the set of admissible values $B(x, t)$ is such that for all $p, q \in B(x, t)$

$$\psi(x, p) - \psi(x, q) - (p - q) \cdot \psi_z(x, q) \geqslant 0, \tag{2.8}$$

where

$$\psi(x, z) = \nu(x) \cdot (\psi^1(z), \ldots, \psi^m(z))^T. \tag{2.9}$$

Remarks. This is the same assumption on the boundary conditions, differently stated, as that needed in Diperna's proof [2] of uniqueness of sufficiently smooth solutions. For $x \in \partial\Omega$, $t > 0$ such that $B(x, t)$ is of small diameter in $\mathbb{R}^n$ and $\partial\Omega$ noncharacteristic (equivalent to $\psi_{zz}(x, q)$ nonsingular for all $q \in B(x, t)$), such a dissipative boundary condition corresponds to specifying the amplitudes of incoming characteristics, in such a manner that the net flow of energy into Ω is nonpositive.

III. MAIN RESULTS

From (1.1), constants may be added to ϕ_{z_j}, ψ_{z_j} without loss of generality. We exploit this flexibility, using the assumption of a solution assuming values in a bounded region $D \subset \mathbb{R}^n$ of phase space, to assume the ϕ_{zj} bounded away from zero in $L_\infty(\Omega)$.

Then there exist constants $\underline{c}$, V_j, W_j such that for all $p \in D$

$$0 < \underline{c} \leqslant \phi_{z_j}(p) \leqslant W_j, \tag{3.1}$$

$$|v_{ij}(p)| \leqslant V_j, \qquad j = 1, \ldots, n, \tag{3.2}$$

where

$$v_{ij}(p) = \psi^i_{z_j}(p)/\phi_{z_j}(p), \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n. \tag{3.3}$$

Our result is stated in the following:

THEOREM. *Let z be any weak solution of* (1.1, 1.2) *corresponding to initial data $z(\cdot, 0) \in E(\Omega)$, assuming values in D, satisfying dissipative boundary conditions and a compatability condition on the initial data at the boundary, of the form*

$$\int_{\partial\Omega} \operatorname*{ess\,sup}_{\substack{|y-x|<\varepsilon \\ y \in \bar{\Omega}}} |z_j(y, 0) - z_j(x, 0)| \, dx \leqslant \varepsilon L_j, \qquad 0 < \varepsilon \leqslant \bar{\varepsilon}, \quad j = 1, \ldots, n, \tag{3.4}$$

for suitable constants L_j. *Suppose further that the initial data satisfies the boundary conditions at later times, i.e.,*

$$z(x,0)\in B(x,t), \qquad x\in\partial\Omega, \quad t>0. \tag{3.5}$$

Then the entropy generation satisfies the following lower bound:

$$G(t)\geqslant -t\left[\frac{1}{\bar{\varepsilon}}\sum_{j=1}^{n} V_j\,\|\phi_{z_j}(z(\cdot,0))\|_{L_\infty(\Omega)}\,|z_j(\cdot,0)|_{E(\Omega)}+\int_{\partial\Omega}\psi(z(\cdot,0))\right]$$
$$-t^2\sum_{j=1}^{n} W_j L_j V_j^2. \tag{3.6}$$

Several remarks precede the proof. First, for constant initial data, $|z(\cdot,0)|_{E(\Omega)}=0$, $\psi^i(z(\cdot,0))$ are constant and the $L_j=0$, so $G(t)\geqslant 0$ from (3.6) as expected. More generally we do not expect the estimate (3.6) to be sharp. Depending on the weak solution z, the function G may not be continuous, in which case (3.6) is understood to hold weakly with respect to t.

The leading term in (3.6), i.e., the term linear in t, depends on the initial data and on V_j, but not on W_j. Interpreting the system (1.1) as a system of particles of local density ϕ_{z_j} moving with speed $v_{\cdot j}$, to first order we see that the entropy generation depends on the speed of the particles, but not on changes in density.

The assumptions of dissipative boundary conditions, (3.4) and (3.5) are used here to make the "boundary contribution" to (3.6) of order t^2. Some such assumptions are needed to prevent a shock from entering the region Ω immediately from the boundary; without these assumptions, one still gets an estimate $G(t)\geqslant -O(t)$, by an easy simplification of the proof given below. For periodic boundary conditions, of course, these assumptions are not needed and the second and third right-hand terms in (3.6) disappear.

The proof of the theorem depends on a technical lemma, effectively concerning the solution of the adjoint problem for (1.1), which is severely complicated by the lack of regularity of the solution z.

LEMMA 2. *For fixed* $t_0>0$, *any* $\tilde{V}_j>V_j, j=1,\dots,n$, *there exist* $\zeta_j\colon\bar{\Omega}\times[0,t_0]\to\mathbb{R}$ *satisfying*

$$\operatorname*{ess\,inf}_{|y-x|\leqslant\tilde{V}_j t_0} z_j(y,0)\leqslant\zeta_j(x,t)$$
$$\leqslant \operatorname*{ess\,sup}_{|y-x|\leqslant\tilde{V}_j t_0} z_j(y,0) \qquad x\in\bar{\Omega},\quad 0\leqslant t\leqslant t_0 \tag{3.7}$$

$$\int_{\Omega} \sum_{i=1}^{n} (\phi_{z_j}(z(\cdot, t_0))\, z_j(\cdot, 0) - \phi_{z_j}(z(\cdot, 0))\, \zeta_j(\cdot, 0))$$

$$+ \int_{\Sigma_{t_0}} \sum_{i=1}^{n} \phi_{z_j} \zeta_j \nu \cdot v_{\cdot j} = 0. \tag{3.8}$$

This proof is given in Section V.

Proof of Theorem. Fix $t > 0$, and recall the entropy density U from (1.4) and the normal entropy flux on $\partial\Omega$ from (1.6), (1.7). Then

$$G(t) = \int_{\Omega} U(z(\cdot, t)) - U(z(\cdot, 0)) + \int_{\Sigma_t} F(z)$$

$$= \int_{\Omega} U(z(\cdot, t)) - U(z(\cdot, 0)) - z(\cdot, 0) \cdot (\phi_z(z(\cdot, t)) - \phi_z(z(\cdot, 0)))$$

$$+ \int_{\Omega} z(\cdot, 0) \cdot (\phi_z(z(\cdot, t)) - \phi_z(z(\cdot, 0))) + \int_{\Sigma_t} F(z)$$

$$\geqslant \int_{\Omega} z(\cdot, 0) \cdot (\phi_z(z(\cdot, t)) - \phi_z(z(\cdot, 0))) + \int_{\Sigma_t} F(z) \tag{3.9}$$

using the convexity of U as recalled in (1.4). Using Lemma 2 in the first term of (3.9), and using (1.6) and (2.9) in the last term, we proceed with

$$G(t) \leqslant \int_{\Omega} \phi_z(z(\cdot, 0)) \cdot (\zeta(\cdot, 0) - z(\cdot, 0)) + \int_{\Sigma_t} z \cdot \psi_z - \psi - \sum_{j=1}^{n} \zeta_j \phi_{z_j} \nu \cdot v_{\cdot j}$$

$$= \int_{\Omega} \phi_z(z(\cdot, 0)) \cdot (\zeta(\cdot, 0) - z(\cdot, 0)) + \int_{\Sigma_t} z \cdot \psi_z - \psi - \zeta \cdot \psi_z \tag{3.10}$$

using the definition of v_{ij} (3.3), and (2.9) again. The boundary term in (3.10) is of $O(t)$, but can be reduced further using the assumptions of dissipative boundary conditions and (3.5), and we continue with

$$G(t) \geqslant \int_{\Omega} \phi_z(z(\cdot, 0)) \cdot (\zeta(\cdot, 0) - z(\cdot, 0))$$

$$+ \int_{\Sigma_t} \psi(z(\cdot, 0)) - \psi(z) - (z(\cdot, 0) - z) \cdot \psi_z(z)$$

$$- \int_{\Sigma_t} \psi(z(\cdot, 0)) + \int_{\Sigma_t} \psi_z(z) \cdot (z(\cdot, 0) - \zeta)$$

$$\geqslant \int_\Omega \phi_z(z(\cdot,0))\cdot(\zeta(\cdot,0)-z(\cdot,0))$$

$$-t\int_{\partial\Omega}\psi(z(\cdot,0))+\int_{\Sigma_t}\psi_z\cdot(z(\cdot,0)-\zeta)$$

$$\geqslant -\sum_{j=1}^n \|\phi_{z_j}(z(\cdot,0))\|_{L_\infty(\Omega)}\int_\Omega |\zeta_j(\cdot,0)-z_j(\cdot,0)|-t\int_{\partial\Omega}\psi(z(\cdot,0))$$

$$-t\sum_{j=1}^n \|\psi_{z_j}\|_{L_\infty(\Sigma_t)}\int_{\partial\Omega}\sup_{0<s\leqslant t}|z_j(\cdot,0)-\zeta_j(\cdot,s)|. \tag{3.11}$$

We use (3.7) for both of the $\zeta_j - z_j(\cdot,0)$ terms in (3.11), using (2.1) with $u=z(\cdot,0)$ in the first term and (3.4) in the second, $\varepsilon=\tilde V_j t$ in both cases, obtaining

$$G(t)\geqslant -\sum_{j=1}^n \|\phi_{z_j}(z(\cdot,0))\|_{L_\infty(\Omega)}\frac{t\tilde V_j}{\bar\varepsilon}|z_j(\cdot,0)|_{E(\Omega)}-t\int_{\partial\Omega}\psi(z(\cdot,0))$$

$$-t\sum_{j=1}^n (W_jV_j)(t\tilde V_j L_j)$$

using also (3.1), (3.2) in the last term. Finally we can replace $\tilde V_j$ by V_j, obtaining (3.6).

IV. PROOF OF LEMMA 1

Suppose $u\in E(\Omega)$ has an essential maximum at y. Denote by b_y the set $\Omega\cap\{x \mid |x-y|<\bar\varepsilon\}$ and μ_y the measure of b_y. As $\partial\Omega$ is smooth, μ_y is bounded away from zero with respect to y, and

$$\mu_y u(y)=\int_{b_y}(u(y)-u(x))\,dx+\int_{b_y}u$$

$$\leqslant |u|_{E(\Omega)}+\|u\|_{L_1(\Omega)}. \tag{4.1}$$

For $q\in E(\Omega)$, (2.4) follows from (4.1) and

$$|qu|_{E(\Omega)}\leqslant |q|_{E(\Omega)}\,\|u\|_{L_\infty(\Omega)}+\|q\|_{L_\infty(\Omega)}\,|u|_{E(\Omega)}. \tag{4.2}$$

To obtain (2.5), it suffices to show that for $u\in E(\Omega)$

$$|\hat u(\xi)|\leqslant c/(1+|\xi|), \tag{4.3}$$

where ^ denotes Fourier transform and ξ is the Fourier transform variable. Extend u as zero outside Ω, and for any $y \in \mathbb{R}^m$, $0 < |y| \leqslant \bar{\varepsilon}$, $\theta \in C(\mathbb{R}^m)$, bounded, we consider the integral

$$
\begin{aligned}
I &= \int_{\Omega} u(x) \frac{\theta(x+y) - \theta(x)}{|y|} \, dx \\
&= \int_{\mathbb{R}^m} (u(x-y) - u(x)) \frac{\theta(x)}{|y|} \, dx \\
&\leqslant \frac{1}{\bar{\varepsilon}} |u|_E \, \|\theta\|_{L_\infty} \\
&\leqslant \frac{1}{\bar{\varepsilon}} |u|_E \, \|\hat{\theta}\|_{L_1} \qquad (4.4)
\end{aligned}
$$

uniformly in $|y|$.

Alternatively

$$
\begin{aligned}
I &= \int_{\mathbb{R}^m} \hat{u}(\xi) \frac{e^{-iy \cdot \xi} - 1}{|y|} \bar{\hat{\theta}}(\xi) \, d\xi \\
&= -i \int_{\mathbb{R}^m} \hat{u}(\xi) \frac{y \cdot \xi}{|y|} \bar{\hat{\theta}}(\xi) \, d\xi + \int_{\mathbb{R}^m} \hat{u} O(y\, |\xi|^2) \bar{\hat{\theta}} \, d\xi. \qquad (4.5)
\end{aligned}
$$

For any fixed $\xi_0 \neq 0$, choosing $y = |y| \, \xi_0 / |\xi_0|$, $|y|$ sufficiently small and $\hat{\theta}$ with its support in a sufficiently small neighborhood of ξ_0, the magnitude of (4.5) is made arbitrarily close to $|\hat{u}(\xi_0) \, \xi_0| \, \|\hat{\theta}\|_{L_1}$. Comparison with (4.4) now establishes (4.3), as $|\hat{u}|$ is bounded in any finite region of $\mathbb{R}^m$.

For given u, let w, η, δ be as in (2.6). Then

$$
\begin{aligned}
\|w - u\|_{L_1(\Omega)} &\leqslant \int_{\Omega} \int \eta(y, x) \, |u(y) - u(x)| \, dx \, dy \\
&\leqslant \int_{\Omega} \int \eta(y, x) \, dx \sup_{|p-y| < \delta} |u(y) - u(p)| \, dy \\
&\leqslant \frac{\delta}{\bar{\varepsilon}} |u|_{E(\Omega)}. \qquad (4.6)
\end{aligned}
$$

Given a sequence u_i, $i = 1, ...,$ bounded in $E(\Omega)$ and weakly convergent, for any fixed $\delta > 0$ determine the corresponding sequence $\{w_i\}$ as in (2.6). In view of (4.6), to obtain (2.7) it suffices to show that the sequence $\{w_i\}$ converges in $L_1(\Omega)$.

Setting

$$\kappa_k(y) = \operatorname*{lub}_{i,\, j>k} \, w_i(y) - w_j(y)$$

$$= \operatorname*{lub}_{i,\, j>k} \int_\Omega \eta(y, x)(u_i(x) - u_j(x))\, dx \tag{4.7}$$

it follows that κ_k is uniformly bounded, nonnegative, and approaches zero pointwise as $k \to \infty$, by weak convergence of $\{u_i\}$. Therefore κ_k approaches zero in $L_1(\Omega)$ as $k \to \infty$, as needed.

V. PROOF OF LEMMA 2

Fix $t_0 > 0$ (presumably small) and j arbitrarily. Where no ambiguity is possible we drop j-subscripts below, abbreviating

$$u = \phi_{z_j}, v_i = v_{ij}, f^i = \psi^i_{z_j} = uv_i, \qquad i = 1, ..., m, \quad V = \tilde{V}_j \tag{5.1}$$

so that within $\Omega_{t_0} = \Omega \times (0, t_0)$

$$u_t + \sum_{i=1}^m (uv_i)_{x_i} = 0 \tag{5.2}$$

in the sense of distributions. However, we cannot obtain ζ simply from the adjoint equation $\zeta_t + \sum_i v_i \zeta_{x_i} = 0$ because of the lack of regularity of the v_i. Of course the difficulty is resolved by smoothing and passing to the limit; it is by this process that we obtain the specific measure of variation of $z(\cdot, 0)$ appearing in the main result (3.6).

First we extend u, f to a region $\tilde{\Omega} = \{(x, t) \mid \operatorname{dist}((x, t), \Omega_{t_0}) < Vt_0\}$ in two steps. First for $0 \leqslant t \leqslant t_0, 0 < \operatorname{dist}(x, \Omega) < Vt_0$, denote by $P(x)$ the (unique) closest point in $\partial\Omega$ to x. On the ray from $P(x)$ through x, we extend $v(P(x)) \cdot f$ as an odd function of $x - P(x)$, while u and the components of f parallel to $\partial\Omega$ at x are extended as even functions,

$$\begin{aligned} &v(P(x)) \cdot f(x, t) \\ &\qquad = 2v(P(x)) \cdot f(P(x), t) - v(P(x)) \cdot f(x', t) \end{aligned} \tag{5.3}$$

$$\begin{aligned} &f(x, t) - v(P(x))(v(P(x)) \cdot f(x, t)) \\ &\qquad = f(x', t) - v(P(x))(v(P(x)) \cdot f(x', t)) \end{aligned} \tag{5.4}$$

$$u(x, t) = u(x', t), \qquad \text{with} \quad x' = 2P(x) - x \in \Omega. \tag{5.5}$$

Next for $t<0$ (resp. $t>t_0$), $\mathrm{dist}(x,\Omega)<Vt_0$ (including $x\in\Omega$), we extend f as an even function and u as an odd function of t (resp $t-t_0$)

$$f(x,t)=f(x,-t),\ u(x,t)=2u(x,0)-u(x,-t),\quad t<0 \tag{5.6}$$

$$f(x,t)=f(x,2t_0-t),\ u(x,t)=2u(x,t_0)-u(x,2t_0-t),\quad t>t_0. \tag{5.7}$$

This extension maintains L_∞ bounds for f and u, but the positive lower bound for u is lost for $t<0$, $t>t_0$. The residual

$$r=u_t+\sum_{i=1}^{m} f^i_{x_i},\qquad (x,t)\in\tilde{\Omega} \tag{5.8}$$

vanishes for $x\in\Omega$, any t, but not for x outside Ω, because of the curvature of $\partial\Omega$.

Let $\rho\colon\mathbb{R}^{m+1}\to\mathbb{R}$ be a smooth function of unit mass and compact support in the unit ball $|s|<1$, depending only on $|s|$ and nonincreasing in $|s|$, with $\rho_\delta(s)=\delta^{-1-m}\rho(s/\delta)$, $\delta>0$, $s\in\mathbb{R}^{m+1}$. From u, f, r as extended to $\tilde{\Omega}$, we obtain

$$u_\delta=\rho_\delta * u,\ f_\delta=\rho_\delta * f,\ r_\delta=\rho_\delta * r \tag{5.9}$$

within $\bar{\Omega}_{t_0}$, e.g.,

$$u_\delta(x,t)=\int_{\tilde{\Omega}}\rho_\delta(x-y,V(t-t'))\,u(y,t')\,dy\,dt',\qquad (x,t)\in\bar{\Omega}_{t_0} \tag{5.10}$$

and then obtain

$$v_\delta(x,t)=f_\delta(z,t)/u_\delta(x,t),\ (x,t)\in\bar{\Omega}_{t_0}. \tag{5.11}$$

From (3.1, 5.1, 5.9, 5.10), we have u_δ, f_δ uniformly bounded, and a uniform positive lower bound for u_δ all with respect to x, t, δ. From (5.11), v_δ is smooth and satisfies $|v_\delta|\leqslant V$ except for x such that $\mathrm{dist}(x,\partial\Omega)<\delta$, where $|v\cdot v_\delta|\leqslant 3V+O(\delta)$. From (5.6), (5.10), we see that for any $x\in\Omega$, $u_\delta(x,0)$ depends only on $u(\cdot,0)$ and is obtained as in (2.6), so that in particular

$$\|u(\cdot,0)-u_\delta(\cdot,0)\|_{L_1(\Omega)}\leqslant c\,\delta|u(\cdot,0)|_{E(\Omega)} \tag{5.12}$$

and an entirely similar statement with $t=t_0$.

Finally, r_δ vanishes except for $\mathrm{dist}(x,\partial\Omega)<\delta$ and satisfies

$$r_\delta(\tilde{x},\tilde{t})|\leqslant c, \tag{5.13}$$

independently of δ, $(\tilde{x},\tilde{t})\in\Omega_{t_0}$, the proof of which is deferred in the interest of continuity.

Next determine $x_\delta(t, x)$, $(x, t) \in \Omega_{t_0}$ from

$$\frac{dx_\delta(t, x)}{dt} = v_\delta(x_\delta(t, x), t),\ x_\delta(t, x) \in \Omega, \qquad 0 < t < t_0 \tag{5.14}$$

$$x_\delta(t_0, x) = x, \qquad x \in \Omega \tag{5.15}$$

and $x_\delta(t, x) = x$ also at any point $(x, t) \in \Sigma_{t_0}$ where no solution of (5.14) forward in time exists remaining within Ω. Setting

$$\zeta_\delta(x_\delta(t, x), t) = z(x, 0) \tag{5.16}$$

it follows that ζ_δ satisfies

$$\zeta_{\delta, t} + v_\delta \zeta_{\delta, x} = 0, \qquad (x, t) \in \Omega_{t_0} \tag{5.17}$$

$$\zeta_\delta(x, t_0) = z(x, t_0), \qquad x \in \Omega \tag{5.18}$$

and as

$$|x_\delta(t, x) - x| \leqslant (V + O(\delta))(t_0 - t), \qquad (x, t) \in \Omega_{t_0} \tag{5.19}$$

we have

$$\begin{aligned} \operatorname*{ess\,inf}_{|y - x| \leqslant Vt_0 + O(\delta)} z(y, 0) &< \zeta_\delta(x, t) \\ &< \operatorname*{ess\,sup}_{|y - x| \leqslant Vt_0 + O(\delta)} z(y, 0), \qquad (x, t) \in \Omega_{t_0}. \end{aligned} \tag{5.20}$$

From (5.8) and (5.9),

$$u_{\delta, t} + f_{\delta, x} = r_\delta. \tag{5.21}$$

Multiplying by ζ_δ, summing over j and integrating by parts in the usual manner, using (5.17) and (5.18) we obtain

$$\begin{aligned} &\int_\Omega \sum_j (u_{j\delta}(\cdot, t_0)\, z_j(\cdot, 0) - u_{j\delta}(\cdot, 0)\, \zeta_{j\delta}(\cdot, 0)) + \int_{\Sigma_{t_0}} \sum_j \zeta_{j\delta}(v \cdot f_{j\delta}) \\ &\quad = \int_{\Omega_{t_0}} \sum_j \zeta_{j\delta}\, r_{j\delta} \end{aligned} \tag{5.22}$$

and equivalently

$$\int_{\Omega} \sum_{j} (u_j(\cdot, t_0)\, z_j(\cdot, 0) - u_j(\cdot, 0)\, \zeta_{j\delta}(\cdot, 0)) + \int_{\Sigma_{t_0}} \sum_{j} \zeta_{j\delta}(\nu \cdot f_{j\delta})$$

$$= \int_{\Omega_{t_0}} \sum_{j} \zeta_{j\delta}\, r_{j\delta} + \int_{\Omega} \sum_{j} ((u_j(\cdot, t_0) - u_{j\delta}(\cdot, t_0))\, z_j(\cdot, 0)$$

$$- (u_j(\cdot, 0) - u_{j\delta}(\cdot, 0))\, \zeta_{j\delta}(\cdot, 0)). \tag{5.23}$$

In (5.23), $|\zeta_{j\delta}|$ is uniformly bounded, so the first right-hand term is $O(\delta)$ from (5.13) and the second is $O(\delta)$ from (5.12) and the corresponding statement at $t = t_0$, since the $z_j(\cdot, 0) \in E(\Omega)$ imply $u_j(\cdot, 0) = \phi_{z_j}(z(\cdot, 0)) \in E(\Omega)$. Thus the right side of (5.23) is of $O(\delta)$

By the smoothness of $\partial\Omega$ and the L_∞ boundedness of f, it follows that for any $(x, t) \in \Sigma_{t_0}$

$$\nu(x) \cdot f_{j\delta}(x, t) = \nu(x) \cdot (\rho_\delta * f_j)(x, t)$$

$$= \int_{\Sigma_{t_0}} \tau(x, t, y, t')\, \nu(y) \cdot f_j(y, t')\, dy\, dt' + O(\delta), \tag{5.24}$$

where τ is nonnegative, of unit mass (in Σ_{t_0}) and has support in $|x - y|^2 + V^2(t - t')^2 < \delta$. Thus the boundary integral in (5.23) satisfies

$$\int_{\Sigma_{t_0}} \sum_{j} \zeta_{j\delta}(\nu \cdot f_{j\delta}) = \int_{\Sigma_{t_0}} \sum_{j} (\nu \cdot f_j)\, \chi_j + O(\delta) \tag{5.25}$$

with

$$\chi_j(y, t') = \int_{\Sigma_{t_0}} \tau(x, t, y, t')\, \zeta_{j\delta}(x, t)\, dx\, dt, \tag{5.26}$$

and in view of (5.20)

$$\operatorname*{ess\,inf}_{\substack{|x-y| \leqslant \tilde{V}_j t_0 + O(\delta) \\ x \in \Omega}} z_j(x, 0) - O(\delta) < \chi_j(y, t')$$

$$< \operatorname*{ess\,sup}_{\substack{|x-y| < \tilde{V}_j t_0 + O(\delta) \\ x \in \Omega}} z_j(x, 0) + O(\delta),$$

$$y \in \partial\Omega, \qquad \delta/V_j < t < t_0 - \delta/V_j, \qquad j = 1, ..., n. \tag{5.27}$$

Thus from (5.24)–(5.27), we have

$$\int_{\Sigma_{t_0}} \sum_j v(x) \cdot f_j(x, t) \operatorname*{ess\,inf}_{|y-x| \leqslant \tilde{V}_j t_0} z_j(y, 0)\, dx\, dt$$

$$\leqslant \liminf_{\delta \downarrow 0} \int_{\Sigma_{t_0}} \sum_j \zeta_{j\delta}\, (v \cdot f_{j\delta}) \tag{5.28}$$

and

$$\limsup_{\delta \downarrow 0} \int_{\Sigma_{t_0}} \sum_j \zeta_{j\delta}\, (v \cdot f_{j\delta})$$

$$\leqslant \int_{\Sigma_{t_0}} \sum_j v(x) \cdot f_j(x, t) \operatorname*{ess\,sup}_{|y-x| \leqslant \tilde{V}_j t_0} z_j(y, 0)\, dx\, dt, \tag{5.29}$$

with ess inf and ess sup interchanged for (x, t) such that $v(x) \cdot f_j(x, t) < 0$. In addition from (5.20)

$$\int_{\Omega} \sum_j u_j(x, 0) \operatorname*{ess\,inf}_{|y-x| \leqslant \tilde{V}_j t_0} z_j(y, 0)\, dx \leqslant \liminf_{\delta \downarrow 0} \int_{\Omega} \sum_j u_j(\cdot, 0)\, \zeta_{j\delta}(\cdot, 0) \tag{5.30}$$

and

$$\limsup_{\delta \downarrow 0} \int_{\Omega} \sum_j u_j(\cdot, 0)\, \zeta_{j\delta}(\cdot, 0) \leqslant \int_{\Omega} \sum_j u_j(x, 0) \operatorname*{ess\,sup}_{|y-x| \leqslant \tilde{V}_j t_0} z_j(y, 0)\, dx. \tag{5.31}$$

Considering the limit of (5.23) as $\delta \downarrow 0$, using the estimates (5.28), (5.29) for the boundary integral and (5.30), (5.31) in the first left hand term, we obtain (3.7), (3.8).

It remains to obtain (5.13). Fix $(\tilde{x}, \tilde{t}) \in \Omega_{t_0}$ arbitrarily, presumably with $\operatorname{dist}(\tilde{x}, \partial\Omega) < \delta$. Denote by

$$\theta(x, t) = \rho_\delta(x - \tilde{x}, V(t - \tilde{t})), \qquad (x, t) \in \tilde{\Omega} \tag{5.32}$$

$$\omega = \{(x, t) \in \operatorname{supp} \theta,\ x \notin \tilde{\Omega}\} \tag{5.33}$$

$$\sigma = (\operatorname{supp} \theta) \cap (\partial\Omega \times \mathbb{R}) \tag{5.34}$$

$$\omega' = \{(x', t),\ x' = 2P(x) - x,\ (x, t) \in \omega\} \tag{5.35}$$

$$\theta'(x', t) = \theta(x, t) \qquad (x', t) \in \omega'. \tag{5.36}$$

Here ω is that portion of supp θ outside $\bar{\Omega}\times\mathbb{R}$, and ω' is the reflexion of ω through the boundary $\partial\Omega\times\mathbb{R}$ as in (5.3)–(5.5). Then

$$r_\delta(\tilde{x},\tilde{t})=\int_{\tilde{\Omega}_{t_0}}\theta(u_t+f_x)$$

$$=-\int_\sigma \theta v\cdot f-\int_\omega \theta_t u+\theta_x\cdot f \tag{5.38}$$

after partial integration, as r vanishes inside Ω_{t_0}.

We redetermine the space coordinates $x_1, ..., x_m$, taking some point in σ as the origin, with x_m normal to $\partial\Omega$ and $x_1, ..., x_{m-1}$ parallel to $\partial\Omega$. Thus for $x\in\omega$, $x'=2P(x)-x\in\omega'$,

$$\frac{\partial x_i'}{\partial x_j}=\delta_{ij}+O(\delta),\qquad i,j=1, ..., m-1 \tag{5.39}$$

$$\frac{\partial x_i'}{\partial x_m},\ \frac{\partial x_m'}{\partial x_i}=O(\delta),\qquad i=1, ..., m-1,\quad x_m'=-x_m. \tag{5.40}$$

We use (5.3)–(5.5), (5.36), and (5.39)–(5.40) to transform (5.38) into an integral over ω'

$$\begin{aligned}
r_\delta(\tilde{x},\tilde{t})&=-\int_\sigma \theta f^m-\int_\omega\left(\theta_t u+\sum_{i=1}^{m-1}\theta_{x_i}f^i+\theta_{x_m}f^m\right)dx\,dt\\
&=-\int_\sigma \theta f^m-\int_\omega (\theta_t'(x',t)\,u(x',t)\\
&\quad+\sum_{i=1}^{m-1}\theta_{x_i'}'(1+O(\delta))\,f^i(x',t)\\
&\quad+2\theta_{x_m}f^m(P(x),t)+\theta_{x_m'}'f^m(x',t))\,dx\,dt\\
&=-\int_\sigma \theta f^m-\int_{\omega'}\theta_t'(1+O(\delta))\,u\,dx'\,dt-\int_\omega\sum_{i=1}^{m}\theta_{x_i'}'f^i(x',t)\\
&\quad+2\int_\sigma f^m\theta+\int_\omega\sum_{i=1}^{m-1}|\theta_{x_i}|\,O(\delta)\\
&=\int_\sigma \theta f^m-\int_{\omega'}\theta_t'(1+O(\delta))\,u\,dx'\,dt-\int_{\omega'}\sum_{i=1}^{m}\theta_{x_i'}'(1+O(\delta))\,f^i\,dx'\,dt\\
&\quad+\int_{\omega'}\sum_{i=1}^{m-1}|\theta_{x_i'}'|\,O(\delta)\\
&=\int_{\omega'}\left(\sum_{i=1}^{m}|\theta_{x_i'}'|+|\theta_t'|\right)O(\delta),
\end{aligned}\tag{5.41}$$

since $u_t + f_x = 0$ weakly within $\Omega \times \mathbb{R}$, and as $\omega' \subset \Omega \times \mathbb{R}$, θ' is a legal test function. From (5.32) and the form of ρ_δ, the right side of (5.41) is uniformly bounded with respect to δ, $\tilde{x}$, $\tilde{t}$, proving (5.13) and completing the proof of the lemma.

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